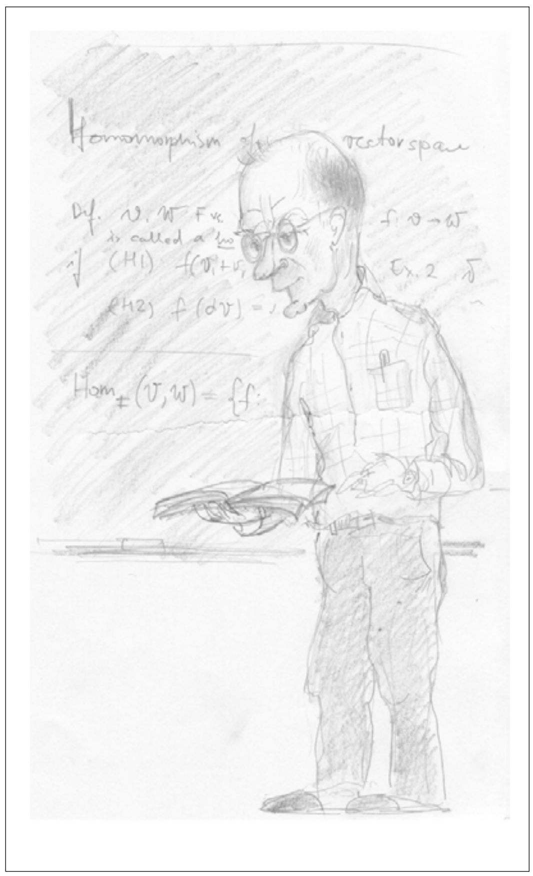
Chapter 11

INTEGRAL EQUATIONS



Chapter 11 Integral Equations

11.1 Normed Vector Spaces
1. Euclidian vector space $\mathbb{R}^n$
2. Vector space of continuous functions $C\left(\bar{G}\right)$
3. Vector Space $L_2\left(G\right)$
4. Cauchy-Bunyakowski inequality
5. Minkowski inequality

11.2 Linear Operators
- continuous operators
- bounded operators
- Lipschitz condition
- contraction operator
- successive approximations
- Banach fixed point theorem

11.3 Integral Operator

11.4 Integral equations
- Fredholm integral equations
- Volterra integral equations
- integro-differential equations
- solution of integral equation

11.5 Solution Methods for Integral Equations
1. Method of successive approximations for Fredholm IE
(Neumann series)
2. Method of successive substitutions for Fredholm IE
(Resolvent method)
3. Method of successive approximations for Volterra IE

11.6 Connection between integral equations and initial and boundary value problems

1. Reduction of IVP to the Volterra IE
2. Reduction of the Volterra IE to IVP
3. Reduction of BVP to the Fredholm IE

11.7 Exercises

<u>Future T o p i c s</u>

11.7 Fixed point theorem (see [Hochstadt “Integral equations”, p.25]) (added in 7.2)

Elementary existence theorems

11.8 Practical applications (see [Jerri “Introduction to Integral Equations with Applications”]) 7.9 Inverse p oblems (see [Jerri, p.17])

11.10 Fredholm’s alternatives

11.1 Normed Vector Spaces

We will start with some definitions and results from the theory of normed vector spaces which will be needed in this chapter (see more details in Chapter 10).

1. Euclidian vector space $\mathbb{R}^n$

The n-dimensional Euclidian vector space consists of all points

$\mathbb{R}^n = \left\{ x = (x_1, x_2, ..., x_n) \middle| x_k \in \mathbb{R} \right\}$

for which the following operations are defined:

Scalar product $\quad (x,y) = x_1 y_1 + x_2 y_2 + ... + x_n y_n \qquad x, y \in \mathbb{R}^n$

Norm $\quad \|x\| = \sqrt{(x,x)} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

Distance $\quad \rho(x,y) = |x - y|$

Convergence $\quad \lim_{k\to\infty} x_k = x$ if $\lim_{k\to\infty} |x - x_k| = 0$

$\mathbb{R}^n$ is a complete vector space (Banach space) relative to defined norm $\|x\|$.

2. Vector space $C(\bar{G})$

Vector space $C(\bar{G})$ consists of all real valued continuous functions defined on the closed domain $\bar{G} \subset \mathbb{R}^n$:

$C(\bar{G}) = \left\{ f(x) : \bar{D} \subset \mathbb{R}^n \to \mathbb{R} \middle| continuous \right\}$

Norm $\quad \|f\|_C = \max_{x\in\bar{G}} |f(x)|$

Convergence $\quad \lim_{k\to\infty} f_k = f$ if $\lim_{k\to\infty} \|f - f_k\|_C = 0$

$C(\bar{G})$ is a complete vector space (Banach space) relative to defined norm $\|f\|_C$.

3. Vector space $L_2(G)$

The space of functions integrable according to Lebesgue (see Section 3.1)

$$L_2(G) = \left\{ f(x) : G \subset \mathbb{R}^n \to \mathbb{C} \middle| \int_G |f(x)|^2 dx < \infty \right\}$$

Inner product $\quad (f,g) = \int_G f(x) \bar{g}(x) dx$

Norm $\quad \|f\|_2 = \sqrt{(f,f)} = \int_G |f(x)|^2 dx$

The following property follows from the definition of the Lebesgue integral

$$\left| \int_G f(x) dx \right| \le \int_G |f(x)| dx$$

$L_2(G)$ is a complete normed vectors spaces (Banach spaces) relative to $\|f\|_2$.

4. Cauchy-Bunyakovsky-Schwarz Inequality (see also Theorem 10.1, p.257)

$$\boxed{|(f,g)| \le \|f\|_2 \cdot \|g\|_2} \qquad \text{for all } f, g \in L_2(G)$$

Proof:

If $f, g \in L_2(G)$, then functions $|f|$, $|g|$ and any combination $\alpha|f| + \beta|g|$ are also integrable and therefore belong to $L_2(G)$.

Consider

$|f| + \lambda|g| \in L_2(G)$, $\lambda \in R$ for which we have

$$0 \le \int_G \left(|f| + \lambda|g|\right)^2 dx = \int_G |f|^2 dx + 2\lambda \int_G |fg| dx + \lambda^2 \int_G |g|^2 dx$$

The right hand side is a quadratic function of λ. Because this function is non-negative, its discriminant ($D = b^2 - 4ac$) is non-positive

$$4\left[\int_G |fg| dx\right]^2 - 4\left(\int_G |f|^2 dx\right)\left(\int_G |g|^2 dx\right) \le 0$$

$$\left[\int_G |fg| dx\right]^2 \le \left(\int_G |f|^2 dx\right)\left(\int_G |g|^2 dx\right)$$

and because $|(f,g)| = \left|\int_G f\bar{g} dx\right| \le \int_G |f\bar{g}| dx \le \int_G |f||g| dx$,

$$|(f,g)|^2 \le \|f\|_2^2 \cdot \|g\|_2^2$$

from which the claimed inequality yields

$$|(f,g)| \le \|f\|_2 \cdot \|g\|_2$$

■

5. Minkowski Inequality (3rd property of the norm "Triangle Inequality"), (see Example 10.7 on p.257)

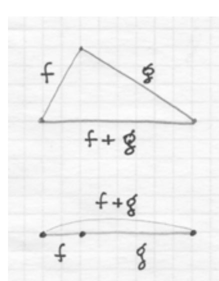
$$\boxed{\|f+g\|_2 \le \|f\|_2 + \|g\|_2} \qquad \text{for all } f,g \in L_2(G)$$

Proof:

Consider
$$\begin{aligned} \|f+g\|_2^2 &= (f+g, f+g) \\ &= (f,f) + (f,g) + (g,f) + (g,g) \\ &\le \|f\|_2^2 + |(f,g)| + |(g,f)| + \|g\|_2^2 \\ &\le \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 \qquad \text{from C-B inequality} \\ &= \left(\|f\|_2 + \|g\|_2\right)^2 \end{aligned}$$

Then extraction of the square root yields the claimed result. ■

Note that the Minkowski inequality reduces to equality only if functions f and g are equal up to the scalar multiple, $f = \alpha g$, $\alpha \in R$ (why?).



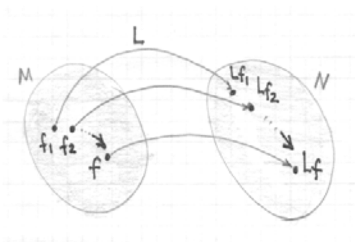
11.2 Linear Operators

Let M and N be two complete normed vectors spaces (Banach spaces, see Ch.10) with norms $\|\cdot\|_M$ and $\|\cdot\|_N$, correspondingly. We define an operator L as a map (function) from the vector space M to the vector space N:

$$L : M \to N$$

Introduce the following definitions concerning the operators in the vector spaces:

Operator $L : M \to N$ is **linear** if $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$ for all $f, g \in M$ and all $\alpha, \beta \in R$



Operator $L : M \to N$ is **continuous** if from $f_k \to f$ in M follows $Lf_k \to Lf$ in N (the image of the convergent sequence in M is a convergent sequence in N)

Operator $L : M \to N$ is **bounded** if there exists $c > 0$ such that $\|Lf\|_N \le c\|f\|_M$ for all $f \in M$

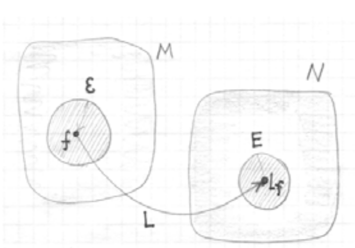
The **norm of operator** $L : M \to N$ can be defined as the greatest lower bound of such constant c

$$\|L\| = \sup_{f \neq 0} \frac{\|Lf\|_N}{\|f\|_M}$$

Theorem 7.1 If linear operator $L : M \to N$ is bounded then it is continuous

Proof:
Let operator $L : M \to N$ be bounded, then according to the definition there exists $c > 0$ such that $\|Lf\|_N \le c\|f\|_M$.

Let $f_k \to f$ in M. That means that $\lim_{k\to\infty}\|f_k - f\|_M = 0$. From the definition of the limit it follows that for any $\varepsilon > 0$ there exists $k_\varepsilon \in N$ such that $\|f_k - f\|_M < \varepsilon$ for all $k \ge k_\varepsilon$.



To prove the theorem, show now that $Lf_k \to Lf$ in N or that $\lim_{k\to\infty}\|Lf_k - Lf\|_N = 0$. We have to show that for any $E > 0$ there exists $K_E \in N$ such that $\|Lf_k - Lf\|_N < E$ for all $k \ge K_E$.

Choose $\varepsilon = \dfrac{E}{c}$, then

$$\|Lf_k - Lf\|_N = \|L(f_k - f)\|_N \le c\|f_k - f\|_M < c \cdot \frac{E}{c} = E \text{ for all } k \ge k_{E/c} = K_E . \ \blacksquare$$

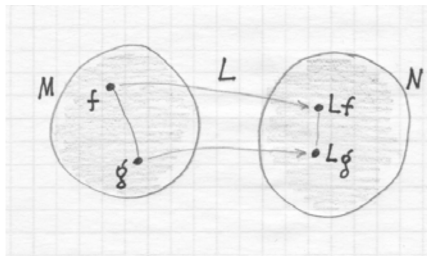
Remark: It is also true that if linear operator is continuous then it is bounded (prove as an exercise). Therefore, for linear operators, properties continuous and bounded are equivalent.

Definition Linear operator $L : M \to N$ satisfies the ***Lipschitz condition*** with constant $k \geq 0$ if

$$\|Lf - Lg\| \leq k\|f - g\| \qquad \text{for all } f, g \in M$$

Obviously that if linear operator satisfies the Lipschitz condition (it is called a Lipschitz operator) then it is bounded (take vector $g = 0$) and, therefore, it is continuous.

Definition Linear operator $L : M \to N$ is a ***contraction*** if it satisfies the Lipschitz condition with constant $k < 1$.



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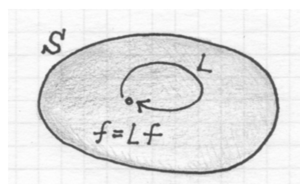
Let S be a closed subset of Banach space M, $S \subset M$, and let $L : S \to S$ be an operator.

Definition Solution of operator equation $f = Lf$ is called a ***fixed point*** of operator L.

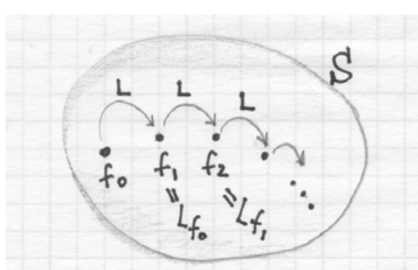


Definition ***Successive approximations*** is a sequence $\{f_0, f_1, f_2, ...\}$ constructed in the following way:

$f_0 \in S$ is a starting point

$f_1 = Lf_0$
$f_2 = Lf_1$
$\vdots$
$f_{n+1} = Lf_n$
$\vdots$

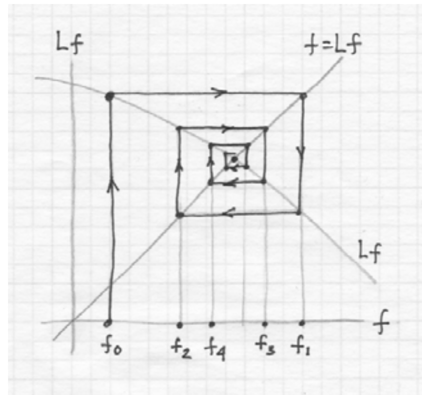
Schematic visualization of successive approximations:



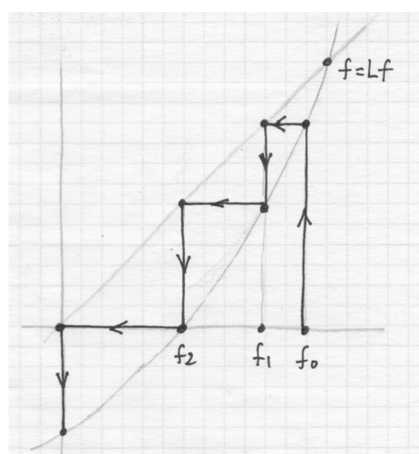
Successive approximations can be used for solution of operator equation

$$f = Lf$$

For example, in this case, the successive approximations converge to the fixed point:

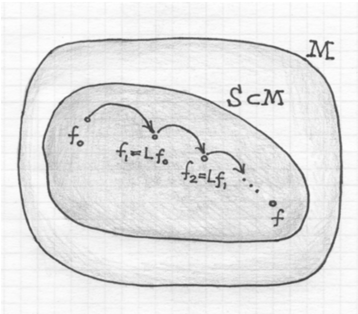


But they do not always converge to the fixed point of operator equation. This example shows that even the choice of the starting point close to the fixed point yields the divergent sequence of successive approximations (apparently they are not very successive ☹):



The following theorem establishes the sufficient condition for convergence of successive approximations to the fixed point of operator equation.

Theorem *(Banach Fixed Point Theorem, 1922)*



Let S be a non-empty closed subset of Banach space M, $S \subset M$, $S \neq \varnothing$.
And let $L : S \to S$ be a contraction operator with constant $k < 1$.
Then the sequence of successive approximations $\left\{ f_n \mid f_n = Lf_{n-1},\, f_0 \in S \right\}$ converges to the unique fixed point $f \in S,\, f = Lf$ for any starting point $f_0 \in S$

$$f_n \to f$$

and the following estimate is valid

$$\|f_n - f\| \le \frac{1}{k}\|f_n - f_{n+1}\| \le \frac{k^n}{1-k}\|f_0 - f_1\|$$

Proof:

- Using mathematical induction, show that $\|f_n - f_{n+1}\| \le k^n \|f_0 - f_1\|$ (☼)

Verify for $n = 0$ $\quad \|f_0 - f_1\| = k^0 \|f_0 - f_1\| = \|f_0 - f_1\|$ *true*

Assume for n: $\quad \|f_n - f_{n+1}\| \le k^n \|f_0 - f_1\|$

Show for $n+1$: $\quad \|f_{n+1} - f_{n+2}\| \le k^{n+1} \|f_0 - f_1\|$

Indeed,

$$\begin{aligned} \|f_{n+1} - f_{n+2}\| &= \|Lf_n - Lf_{n+1}\| && \text{definition of s.a.} \\ &\le k\|f_n - f_{n+1}\| && \text{Lipschitz condition} \\ &\le k \cdot k^n \|f_0 - f_1\| && \text{assumption} \\ &= k^{n+1}\|f_0 - f_1\| \end{aligned}$$

- Show that $\{f_n\}$ is a Cauchy sequence, i.e. $\lim\limits_{n,m\to\infty} \|f_m - f_n\| = 0$

Consider

$$f_m - f_n = f_m - Lf_m + Lf_m - Lf_n + Lf_n - f_n \qquad \text{(add and subtract)}$$

Apply Minkowski inequality twice:

$$\begin{aligned} \|f_m - f_n\| &\le \|f_m - Lf_m\| + \|Lf_m - Lf_n\| + \|Lf_n - f_n\| \\ &\le \|f_m - Lf_m\| + k\|f_m - f_n\| + \|Lf_n - f_n\| && \text{Lipschitz condition} \\ &\le \frac{\|f_m - Lf_m\| + \|Lf_n - f_n\|}{1-k} \\ &\le \frac{\|f_m - f_{m+1}\| + \|f_n - f_{n+1}\|}{1-k} && \text{definition of s.a.} \\ &\le \frac{k^m\|f_0 - f_1\| + k^n\|f_0 - f_1\|}{1-k} && \text{equation (☼)} \\ &= \frac{k^m + k^n}{1-k}\|f_0 - f_1\| \quad \to 0 \quad \text{when } m,n \to \infty \end{aligned}$$

- Because vector space M is complete, Cauchy sequence $\{f_n\}$ converges to some $f \in M$. And because $f_n \in S$ and set S is closed (includes all limiting points), $f \in S$. Therefore in a limit, equation of successive approximations

$$f_{n+1} = Lf_n \quad \Rightarrow \quad \lim_{n\to\infty} f_{n+1} = \lim_{n\to\infty} Lf_n$$

$$\lim_{n\to\infty} f_{n+1} = L\left(\lim_{n\to\infty} f_n\right)$$

converges to

$$f = Lf$$

And therefore, $f \in S$ is a fixed point.

- (Uniqueness) Let $f, g \in S$ be two fixed points of operator L:

$$f = Lf$$
$$g = Lg$$

Then from

$$\|f - g\| = \|Lf - Lg\| \le k\|f - g\|$$

yields

$$(1-k)\|f - g\| \le 0$$

Because $1 - k > 0$

$$\|f - g\| \le 0$$

That is possible only if

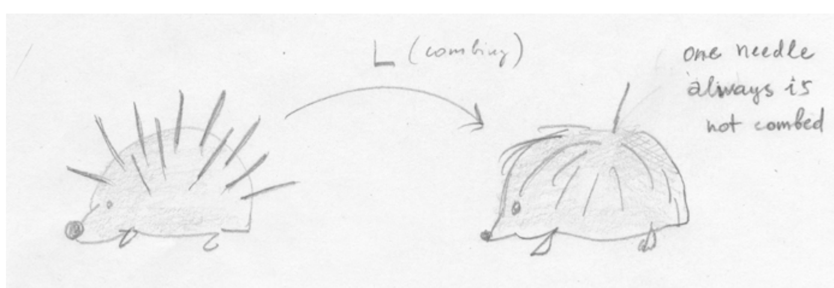
$$\|f - g\| = 0$$

Therefore, $f = g$

So the fixed point is unique. ■

Hugo Steinhaus, the colleague and friend of Stefan Banach, formulated the fixed point theorem in the following way:

"hedgehog cannot be combed"



7.3 Integral Operator

Consider an operator called an **integral operator** given by the equation

$$Kf = \int_G K(x,y) f(y) dy \qquad x \in G \subset R^n$$

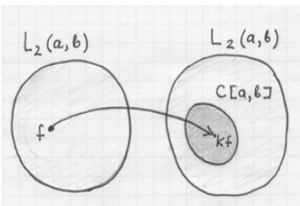
Obviously, that integral operator is linear.

Function $K(x,y)$ is called a **kernel** of the integral operator. We will consider kernels $K(x,y) \in L_2(G \times G)$, therefore

$$\int_G \int_G |K(x,y)|^2 \, dxdy < \infty$$

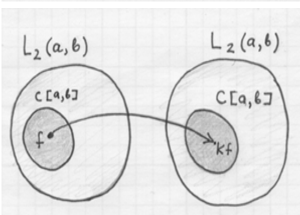
In a case of $G \subset R$, the domain $G = (a,b)$, where a, b can be finite or infinite.

Theorem 7.2 Let K be the integral operator with a kernel $K(x,y)$ continuous in $[a,b] \times [a,b]$. Then operator K is bounded, and, therefore, continuous. Moreover:



1) $K : L_2(a,b) \to C[a,b] \qquad \|Kf\|_C \le M\sqrt{b-a}\,\|f\|_2 \qquad$ for $f \in L_2(a,b)$

2) $K : L_2(a,b) \to L_2(a,b) \qquad \|Kf\|_2 \le M(b-a)\|f\|_2 \qquad$ for $f \in L_2(a,b)$



3) $K : C[a,b] \to C[a,b] \qquad \|Kf\|_C \le M(b-a)\|f\|_C \qquad$ for $f \in C[a,b]$

Proof:
Since $K(x,y)$ is continuous in the closed domain $[a,b] \times [a,b]$, there exists $M > 0$ such that $M = \max\limits_{x,y \in [a,b]} |K(x,y)|$.

1) Let $f \in L_2(a,b)$. Then because function $K(x,y)$ is continuous in $[a,b] \times [a,b]$, the function $(Kf)(x)$ is continuous in $[a,b]$, and, therefore $K : L_2(a,b) \to C[a,b]$. Consider

$$\|Kf\|_C = \max_{x\in[a,b]} |(Kf)(x)| \qquad \text{definition of norm in } C[a,b]$$

$$= \max_{x\in[a,b]} \left| \int_a^b K(x,y) f(y) dy \right| \qquad \text{definition of integral operator}$$

$$= \max_{x\in[a,b]} |(K(x,y), f(y))| \qquad \text{inner product in } L_2(a,b)$$

$$\le \max_{x\in[a,b]} \|K\|_2 \|f\|_2 \qquad \text{Cauchy-Bunyakowski inequality}$$

$$\le \|f\|_2 \max_{x\in[a,b]} \left[\int_a^b |K(x,y)|^2 \, dy \right]^{1/2} \qquad \text{definition of norm in } L_2(a,b)$$

$$\le \|f\|_2 \max_{x\in[a,b]} \left[\int_a^b M^2 dy \right]^{1/2} \qquad \text{replace by } M = \max_{x,y\in[a,b]} |K(x,y)|$$

$$= M\sqrt{b-a}\,\|f\|_2 \qquad \text{calculating definite integral}$$

2) $\|Kf\|_2 \quad = \left((Kf)(x),(Kf)(x)\right)^{1/2}$ definition of norm in $L_2(a,b)$

$= \left[\int_a^b |(Kf)(x)|^2\, dx\right]^{1/2}$ inner product in $L_2(a,b)$

$= \left[\int_a^b \left|\int_a^b K(x,y) f(y) dy\right|^2 dx\right]^{1/2}$ definition of integral operator

$\le \left[\int_a^b \|K\|_2^2 \|f\|_2^2\, dx\right]^{1/2}$ Cauchy-Bunyakowski inequality

$= \|f\|_2 \left[\int_a^b \left(\int_a^b |K(x,y)|^2\, dy\right) dx\right]^{1/2}$ factoring $\|f\|_2$

$\le \|f\|_2 \left[\int_a^b \left(\int_a^b M^2\, dy\right) dx\right]^{1/2}$ replace by $M = \max\limits_{x,y\in[a,b]} |K(x,y)|$

$\le M\|f\|_2 \left[\int_a^b \left(\int_a^b dy\right) dx\right]^{1/2}$ calculating definite integral

$= M\|f\|_2 (b-a)$

3) $\|Kf\|_C \quad = \max\limits_{x\in[a,b]} |(Kf)(x)|$ definition of norm in $C[a,b]$

$= \max\limits_{x\in[a,b]} \left|\int_a^b K(x,y) f(y) dy\right|$ definition of integral operator

$\le \max\limits_{x\in[a,b]} \int_a^b |K(x,y)||f(y)| dy$

$\le \max\limits_{x\in[a,b]} \int_a^b M\, |f(y)| dy$ replace by $M = \max\limits_{x,y\in[a,b]} |K(x,y)|$

$= M \int_a^b |f(y)| dy$ does not depend on x

$\le M \int_a^b \max\limits_{y\in[a,b]} |f(y)| dy$

$\le M \int_a^b \|f\|_C\, dy$ definition of norm in $C[a,b]$

$\le M \|f\|_C \int_a^b dy$

$= M(b-a)\|f\|_C$ calculating definite integral

■

7.4 Integral Equations

Integral equations are equations in which the unknown function is under the integral sign. The typical integral equations for unknown function $u(x)$, $x \in G \subset \mathbb{R}^n$ (in this chapter , we consider $x \in (a,b) \subset \mathbb{R}$) include integral term in the form of integral operator with the kernel $K(x,y)$

$$Ku = \int_a^{b(x)} K(x,y)u(y)\,dy$$

The main types of integral equations are the following:

I Fredholm integral equation

1) Fredholm's integral equation of the 1st kind:

$\int_a^b K(x,y)u(y)\,dy = f(x)$	$Ku = f$	non-homogeneous eqn
$\int_a^b K(x,y)u(y)\,dy = 0$	$Ku = 0$	homogeneous eqn

2) Fredholm's integral equation of the 2nd kind: $\lambda \in C$ is a parameter

$u(x) = \lambda \int_a^b K(x,y)u(y)\,dy + f(x)$	$u = \lambda Ku + f$	non-homogeneous eqn
$u(x) = \lambda \int_a^b K(x,y)u(y)\,dy$	$u = \lambda Ku$	homogeneous eqn

II Volterra integral equation

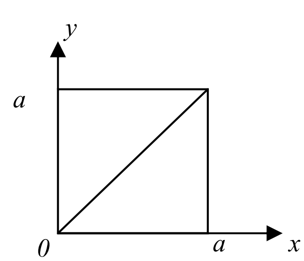
1) Volterra's integral equation of the 1st kind:

$$\int_0^x K(x,y)u(y)\,dy = f(x)$$

2) Volterra's integral equation of the 2nd kind:

$$u(x) = \lambda \int_0^x K(x,y)u(y)\,dy + f(x)$$

Note that Volterra's equations can be viewed as a special case of Fredholm's equations with $K(x,y) = 0$ for $0 < x < y < a$ (it is called a Volterra kernel).



III Integro-Differential Equation includes an unknown function under the integral sign and also any derivative of the unknown function. For example:

$$\frac{du}{dx} = u(x) + \int_G K(x,y)u(y)dy + f(x)$$

An important representation of the integro-differential equation is a Radiative Transfer Equation describing energy transport in the absorbing, emitting and scattering media (analogous equations appear in the theory of neutron transport).

Solution of integral equation is any function $u(x)$ satisfying this equation:

$u = \lambda Ku + f$ non-homogeneous equation

$u = \lambda Ku$ homogeneous equation

The value of the parameter $\lambda \in C$ for which the homogeneous integral equation has a non-trivial solution $u \in L_2$ which is called an **eigenvalue** of the kernel $K(x,y)$, and the corresponding solution is called an **eigenfunction** of this kernel.

Eigenvalue problem We will distinguish eigenvalue problems for the integral kernel (integral equation):

$$u = \lambda Ku$$

and for the integral operator

$$Ku = \frac{1}{\lambda}u$$

The eigenvalues of the integral operator are recipical to eigenvalues of the integral kernel, and eigenfunctions are the same in both cases.

Existence of the solution of Fredholm's integral equation

Consider Fredholm's integral equation of the 2nd kind:

$$u = \lambda Ku + f \qquad (\Diamond)$$

with bounded integral operator K which also satisfies the Lipschitz condition:

$$\|Ku_1 - Ku_2\| \le k\|u_1 - u_2\|, \quad k \ge 0$$

Rewrite integral equation in the form

$$u = Tu \qquad (\Diamond\Diamond)$$

where operator T is defined by

$$Tu = \lambda Ku + f$$

Note that operator T is not linear. Obviously, if u is a fixed point of operator equation (◊◊), then u is a solution of integral equation (◊).

Consider

$$\begin{aligned} \|Tu_1 - Tu_2\| &= \|\lambda Ku_1 + f - (\lambda Ku_2 + f)\| \\ &= \|\lambda Ku_1 - \lambda Ku_2\| \\ &= |\lambda|\|K(u_1 - u_2)\| \\ &\le |\lambda| k \|u_1 - u_2\| \end{aligned}$$

If $|\lambda| k < 1$, then operator T is a contraction and according to Banach fixed point theorem , there exist a unique fixed point of equation (◊◊) . This unique fixed point is also a solution of Fredholm's equation (◊). Therefore, the following conclusion can be made: Fredholm's integral equation of the 2nd kind with bounded kernel has a unique solution for sufficiently small $|\lambda|$, in fact $|\lambda| < 1/k$.

7.5 Solution Methods for Integral Equations

1. The Method of Successive Approximations for Fredholm's Integral Equation

For the integral equation

$$u = \lambda K u + f$$

the following iterations of the method of successive approximations are set by:

$$u_0(x) = f(x)$$

$$u_n(x) = \lambda K u_{n-1} + f \qquad n = 1,2,\ldots$$

Lemma 7.1 $\boxed{u_n(x) = \sum_{k=0}^{n} \lambda^k K^k f}$ where $K^k = \underbrace{K(K(\cdots K))}_{k\ times}$

Proof by mathematical induction (assume that the formula for n is true):

$n = 0 \qquad u_0(x) = \lambda^0 K^0 f = f(x)$ confirmed

$n = n+1 \qquad u_{n+1}(x) = \lambda K u_n + f$ by definition

$= \lambda K\left(\sum_{k=0}^{n} \lambda^k K^k f\right) + f$ by assumption

$= f + \sum_{k=0}^{n} \lambda^{k+1} K^{k+1} f$ linearity

$= f + \sum_{p=1}^{n+1} \lambda^p K^p f$ change of index $p = k+1$

$= \lambda^0 K^0 f + \sum_{p=1}^{n+1} \lambda^p K^p f$

$= \sum_{p=0}^{n+1} \lambda^p K^p f$

$= \sum_{k=0}^{n+1} \lambda^k K^k f$ change of index $p = k$ ■

Neumann Series $\sum_{k=0}^{\infty} \lambda^k K^k f$ is called to be the Neumann Series

Estimation of iterations $\left\|K^n f\right\|_C = \left\|K\left(K^{n-1} f\right)\right\|_C$

$\le M(b-a)\left\|K^{n-1} f\right\|_C$ Theorem 7.2 (3)

$\le M^2 (b-a)^2 \left\|K^{n-2} f\right\|_C$

$\ldots$

$\le M^n (b-a)^n \left\|f\right\|_C$

$$\left\|\sum_{k=0}^{\infty}\lambda^k K^k f\right\|_C \le \sum_{k=0}^{\infty}\left|\lambda^k\right|\left\|K^k f\right\|_C \qquad \text{Cauchy-Bunyakovsky inequality}$$

$$\le \|f\|_C \sum_{k=0}^{\infty}|\lambda|^k M^k (b-a)^k \qquad \text{Theorem 7.2 3)}$$

$$= \|f\|_C \sum_{k=0}^{\infty}\left[|\lambda| M (b-a)\right]^k \qquad \text{geometric series}$$

converges if $|\lambda| < \dfrac{1}{M(b-a)}$

$$= \frac{\|f\|_C}{1-|\lambda| M(b-a)}$$

Therefore, the Neumann series $\sum_{k=0}^{\infty}\lambda^k K^k f$ converges for $|\lambda| < \dfrac{1}{M(b-a)}$.

Denote the sum of the Neumann series as a function $u(x)$:

$$u(x) = \sum_{k=0}^{\infty}\lambda^k K^k f$$

Show that this function satisfies the integral equation $u = \lambda K u + f$. Consider iterations

$u_n(x) = \lambda K u_{n-1} + f$

then

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lambda K \lim_{n\to\infty} u_{n-1}(x) + f \\ &= \lambda \int_a^b K(x,y) \lim_{n\to\infty} u_{n-1}(y)dy + f \\ &= \lambda \int_a^b K(x,y) u(y) dy + f \end{aligned}$$

And, recalling estimation, $\|u(x)\|_C \le \dfrac{\|f\|_C}{1-|\lambda| M(b-a)}$

show that this solution is unique. For that, it is enough to show that the homogeneous equation $u = \lambda K u$ has only a trivial solution. Indeed, if $u_0 = \lambda K u_0$, then $u_0 \in C[a,b]$ and , according to Theorem 6.2 3),

$\|u_0\|_C \le |\lambda| M(b-a)\|u_0\|_C$, then

$[1-|\lambda| M(b-a)]\|u_0\|_C \le 0$

Because $|\lambda| < \dfrac{1}{M(b-a)}$, $[1-|\lambda| M(b-a)] > 0$ and, therefore, $\|u_0\|_C = 0$. That yields, that $u\ (x) = 0$ for all $x \in [a,b]$. So, only the trivial solution exists for the homogeneous equation.

The non-homogeneous equation $u = \lambda K u + f$ can be rewritten in the form

$(I - \lambda K)u = f$

where I is an identity operator

Then solution of this equation can be treated as an inversion of the operator

$u = (I - \lambda K)^{-1} f$

Therefore, if $|\lambda| < \dfrac{1}{M(b-a)}$, then there exists an inverse operator $(I - \lambda K)^{-1}$.

The above mentioned results can be formulated in the following theorem:

Theorem 7.3 Fredholm's integral equation

$$u = \lambda K u + f$$

with $|\lambda| < \dfrac{1}{M(b-a)}$ and continuous kernel $K(x,y)$ has a unique solution $u(x) \in C[a,b]$ for any $f(x) \in C[a,b]$.
This solution is given by a convergent Neumann series

$$u(x) = \sum_{k=0}^{\infty} \lambda^k K^k f$$

and satisfies

$$\|u(x)\|_C \le \frac{\|f\|_C}{1 - |\lambda| M(b-a)}.$$

If $|\lambda| < \dfrac{1}{M(b-a)}$, then there exists an inverse operator $(I - \lambda K)^{-1}$.

Conditions of Theorem 7.3 are only just sufficient conditions; if these conditions are not satisfied, solution of the integral equation still can exists and the Neumann series can be convergent.

Example 7.1 Find the solution of the integral equation

$$u(x) = e^x + \frac{1}{e}\int_0^1 u(y)\,dy$$

by the method of successive approximations and in the form of the Neumann series.

Identify: $K(x,y) = 1$ $\quad f(x) = e^x$ $\quad b - a = 1$

$M = 1$ $\quad \lambda = \dfrac{1}{e}$

Check condition: $|\lambda| < \dfrac{1}{M(b-a)}$ $\quad \dfrac{1}{e} < \dfrac{1}{1 \cdot 1} < 1$

1) iterations:

$$u_0(x) = e^x$$

$$u_1(x) = e^x + \frac{1}{e}\int_0^1 u_0(y)\,dy = e^x + \frac{1}{e}\int_0^1 e^x\,dy = e^x + \frac{1}{e}\left[e^x\right]_0^1 = e^x + 1 - \frac{1}{e}$$

$$u_2(x) = e^x + \frac{1}{e}\int_0^1 u_1(y)\,dy = e^x + \frac{1}{e}\int_0^1 \left(e^x + 1 - \frac{1}{e}\right)dy = e^x + 1 - \frac{1}{e^2}$$

…

$$u_n(x) = e^x + \frac{1}{e}\int_0^1 u_{n-1}(y)\,dy = e^x + 1 - \frac{1}{e^n}$$

Then solution of the integral equation is a limit of iterations

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left(e^x + 1 - \frac{1}{e^n}\right) = e^x + 1$$

This result can be validated by a direct substitution into the given integral equation.

2) Neumann series:

$$u(x)=\sum_{k=0}^{\infty}\lambda^k K^k f \qquad = f(x)+\lambda^1 K^1 f+\lambda^2 K^2 f+\cdots$$

$$\begin{aligned}
f(x) &= e^x \\
Kf &= \int_0^1 e^x\,dy &&= e-1 \\
K^2 f &= \int_0^1 (e-1)\,dy &&= e-1 \\
&\cdots \\
K^n f &= e-1
\end{aligned}$$

Then the Neumann series is

$$\begin{aligned}
u(x) &= e^x+\frac{1}{e}(e-1)+\frac{1}{e^2}(e-1)+\cdots+\frac{1}{e^n}(e-1)+\cdots \\
&= e^x-(e-1)+(e-1)+\frac{1}{e}(e-1)+\frac{1}{e^2}(e-1)+\cdots+\frac{1}{e^n}(e-1)+\cdots \\
&= e^x-(e-1)+(e-1)\sum_{n=0}^{\infty}\frac{1}{e^n} \\
&= e^x-e+1+\frac{(e-1)}{1-\frac{1}{e}} \\
&= e^x-e+1+e \\
&= e^x+1
\end{aligned}$$

So, the Neumann series approach produces the same solution.

2. The Method of Successive Substitutions for Fredholm's Integral Equation (the Resolvent Method)

Iterated kernel

Let integral operator K has a continuous kernel $K(x, y)$, then define:

Repeated operator $\quad K^n = K\left(K^{n-1}\right) = \left(K^{n-1}\right)K \qquad n = 2,3,\ldots$

It has a kernel $\quad \boxed{K_n(x, y) = \int_G K(x, y')K_{n-1}(y', y)dy'}$

Indeed,
$$\left(K^1 f\right)(x) = \int_G \underbrace{K(x, y)}_{K_1(x,y)} f(y)\,dy$$

$$\begin{aligned}\left(K^2 f\right)(x) &= [K(Kf)](x)\\ &= \int_G K(x, y')\left[\int_G K(y', y)f(y)dy\right]dy'\\ &= \int_G \underbrace{\left[\int_G K(x, y')K(y', y)dy'\right]}_{K_2(x,y)} f(y)\,dy\end{aligned}$$

…

Kernel
$$\begin{aligned}K_n(x, y) &= \int_G K(x, y')K_{n-1}(y', y)dy'\\ &= \int_G K_{n-1}(x, y')K(y', y)dy'\end{aligned}$$

is called an **iterated kernel**. Kernels $K_n(x, y)$ are continuous, and if domain $G = (a, b)$, then
$$\left|K_n(x, y)\right| \le M^n (b-a)^{n-1}$$

Resolvent

Function defined by the infinite series
$$R(x, y, \lambda) = \sum_{k=0}^{\infty} \lambda^k K_{k+1}(x, y)$$
is called a **resolvent**.

Theorem 7.4 Solution of integral equation $u = \lambda K u + f$ with continuous kernel $K(x, y)$ is unique in $C[a, b]$ for $|\lambda| < \dfrac{1}{M(b-a)}$, and for any $f \in C[a, b]$ is given by

$$\boxed{u(x) = f(x) + \lambda \int_a^b R(x, y, \lambda) f(y)dy}$$

i.e. there exists inverse operator
$$(I - \lambda K)^{-1} = I + \lambda R, \qquad |\lambda| < \frac{1}{M(b-a)}$$

Example 7.2 Find solution of integral equation

$$u(x)=\frac{23}{6}x+\frac{1}{8}\int_0^1 xyu(y)dy$$

by the resolvent method .

Identify: $K(x,y)=xy$ $\quad f(x)=\frac{23}{6}x$ $\quad b-a=1$

$M=1$ $\quad \lambda=\frac{1}{8}$

Check condition: $|\lambda|<\frac{1}{M(b-a)}$ $\quad \frac{1}{8}<\frac{1}{1\cdot 1}<1$

Iterated kernels:

$$K_1(x,y) = xy$$

$$K_2(x,y) = \int_0^1 K(x,y')K_1(y',y)dy' = \int_0^1 xy'y'ydy' = xy\left[\frac{y'^3}{3}\right]_0^1 = \frac{xy}{3}$$

$$K_3(x,y) = \int_0^1 K_2(x,y')K_2(y',y)dy' = \int_0^1 \frac{xy'}{3}y'ydy' = \frac{xy}{3}\left[\frac{y'^3}{3}\right]_0^1 = \frac{xy}{3^2}$$

…

$$K_n(x,y) = \frac{xy}{3^{n-1}}$$

Resolvent:

$$\begin{aligned}
R(x,y,\lambda) &= \sum_{k=0}^{\infty}\lambda^k K_{k+1}(x,y)\\
&= xy+\frac{1}{8}\frac{xy}{3}+\frac{1}{8^2}\frac{xy}{3^2}+\frac{1}{8^3}\frac{xy}{3^3}+\ldots+\frac{1}{8^n}\frac{xy}{3^n}+\ldots\\
&= xy\left[1+\frac{1}{8}\frac{1}{3}+\frac{1}{8^2}\frac{1}{3^2}+\frac{1}{8^3}\frac{1}{3^3}+\ldots+\frac{1}{8^n}\frac{1}{3^n}+\ldots\right]\\
&= xy\frac{1}{1-\frac{1}{24}}\\
&= \frac{24}{23}xy
\end{aligned}$$

Solution:

$$\begin{aligned}
u(x) &= f(x)+\lambda\int_a^b R(x,y,\lambda)f(y)dy\\
&= \frac{23}{6}x+\frac{1}{8}\int_0^1\frac{24}{23}xy\frac{23}{6}ydy\\
&= \frac{23}{6}x+\frac{1}{2}x\int_0^1 y^2dy\\
&= \frac{23}{6}x+\frac{1}{2}x\left[\frac{y^3}{3}\right]_0^1\\
&= 4x
\end{aligned}$$

3. The Method of Successive Approximations for the Volterra Integral Equation of the 2nd kind

Consider the Volterra integral equation of the 2nd kind

$$u(x) = \lambda \int_0^x K(x,y)\,u(y)\,dy + f(x)$$

where $K(x,y)$ is a continuous kernel, $K(x,y) \in C([a,b]\times[a,b])$.

The method of successive approximation is defined by the following iterations:

$$u_0(x) \quad = f(x)$$

$$u_n(x) \quad = \sum_{k=0}^{n} \lambda^n K^k f \qquad = \lambda K u_{n-1} + f$$

Theorem 7.5 The Volterra integral equation of the 2nd kind

$$u(x) = \lambda \int_0^x K(x,y)\,u(y)\,dy + f(x)$$

with continuous kernel $K(x,y)$ and with any $\lambda \in R$

has a unique solution $u(x) \in C[0,a]$ for any $f(x) \in C[0,a]$. This solution is given by a uniformly convergent Neumann series

$$u(x) = \sum_{k=0}^{\infty} \lambda^n \left(K^k f\right)(x)$$

and its norm satisfies

$$\|u(x)\|_C \le \|f\|_C \, e^{|\lambda| Ma}$$

Example 7.3 Find solution of integral equation

$$u(x) = 1 + \int_0^x u(y)\,dy$$

by the method of successive approximations.

Identify: $K(x,y) = 1$ $\qquad f(x) = 1$

$M = 1$ $\qquad \lambda = 1$

$$K^0 f \quad = f(x) \quad = 1$$

$$K^1 f \quad = \int_0^x K(x,y)\left(K^0 f\right)(y)\,dy \qquad = \int_0^x 1 \cdot 1\,dy \qquad = [y]_0^x \quad = x$$

$$K^2 f \quad = \int_0^x K(x,y)\left(K^1 f\right)(y)\,dy \qquad = \int_0^x 1 \cdot y\,dy \qquad = \left[\frac{y^2}{2}\right]_0^x \qquad = \frac{x^2}{2}$$

$$K^3 f \quad = \int_0^x K(x,y)\left(K^2 f\right)(y)\,dy \qquad = \int_0^x 1 \cdot \frac{y^2}{2}\,dy \quad = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^x \qquad = \frac{x^3}{2 \cdot 3}$$

…

$$K^n f \quad = \frac{x^n}{n!}$$

Solution: $\qquad u(x) \quad = \sum_{k=0}^{\infty} \lambda^n \left(K^k f\right)(x) \qquad = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad = e^x$

7.6 Connection between integral equations and initial and boundary value problems

1. Reduction of IVP to the Volterra integral equation

Example 7.4 Reduce IVP

$$u' - 3x^2 u = 0 \qquad u(0) = 1$$

to the Volterra integral equation.

Integrate the differential equation from 0 to x:

$$\int_0^x \left[u'(y) - 3y^2 u(y)\right] dy = 0$$

$$\int_0^x (u')dy - \int_0^x (3y^2 u) dy = 0$$

$$u(x) - u(0) - 3\int_0^x y^2 u(y) dy = 0$$ use the initial condition $u(0) = 1$

$$u(x) = 1 + 3\int_0^x y^2 u(y) dy$$ is a Volterra equation with $K(x,y) = y^2$

2. Reduction of the Volterra integral equation to IVP

Recall the Liebnitz rule for differentiation of expressions with integrals:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(x,y)dy \quad = \int_{a(x)}^{b(x)} \frac{\partial g(x,y)}{\partial x} dy + g\left[x, b(x)\right]\frac{db(x)}{dx} - g\left[x, a(x)\right]\frac{da(x)}{dx}$$

In particularly,

$$\frac{d}{dx}\int_0^x g(y)dy \quad = g(x)$$

$$\frac{d}{dx}\int_0^x g(x,y)dy \quad = \int_0^x \frac{g(x,y)}{\partial x} dy + g(x,x)$$

Reduction of the Volterra integral equation to IVP is performed by consecutive differentiation of the integral equation with respect to variable x and substitution $x = 0$ for setting of the initial conditions.

Example 7.5 Reduce the Volterra integral equation

$$u(x) = x^3 + \int_0^x (x-y)^2 u(y) dy$$

initial value problem.

substitute $x = 0$ to get initial condition

$$u(x) = x^3 + \int_0^x (x-y)^2 u(y) dy \qquad u(0) = 0^3 + \underbrace{\int_0^0 (x-y)^2 u(y)\, dy}_{0} \quad u(0) = 0$$

$$u'(x)=3x^2+\int_0^x 2(x-y)u(y)dy \qquad u'(0)=30^2+\int_0^0 2(x-y)u(y)dy \quad u'(0)=0$$

$$u''(x)=3x^2+2\int_0^x u(y)dy \qquad u''(0)=30^2+2\int_0^x u(y)dy \qquad u''(0)=0$$

$$u'''(x)=6x+2u(x)$$

Therefore, the integral equation is reduced to IVP for 3rd order ODE:

$$u'''-2u=6x \qquad \begin{aligned} u(0)&=0\\ u'(0)&=0\\ u''(0)&=0\end{aligned}$$

3. Reduction of BVP to the Fredholm integral equation

Recall repeated integration formula:

$$\int_0^x\int_0^{t_n}\cdots\int_0^{t_3}\int_0^{t_2} f(t_1)dt_1dt_2\cdots dt_{n-1}dt_n=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f(t)\,dt$$

Example 7.6 Reduce the boundary value problem

$$y''(x)+y(x)=x \quad x\in(0,\pi)$$
$$y(0)=1$$
$$y(\pi)=\pi-1$$

to the Fredholm integral equation.

Set $\qquad y''(x)=u(x)$

integrate $\qquad \int_0^x y''(t)dt=\int_0^x u(t)dt$

$$y'(x)-y'(0)=\int_0^x u(t)dt$$

integrate $\qquad \int_0^x [y'(t_2)-y'(0)]dt_2=\int_0^x\left[\int_0^{t_2}u(t_1)dt_1\right]dt_2$

$$y(x)-y(0)-y'(0)x=\int_0^x\left[\int_0^{t_2}u(t_1)dt_1\right]dt_2$$

$$y(x)-y(0)-y'(0)x=\int_0^x (x-t)u(t)dt \qquad \text{repeated integration}$$

Use the first boundary condition

$$y(x)=1+y'(0)x+\int_0^x (x-t)u(t)dt$$

In this expression, $y'(0)$ is not known. Substitute $x=\pi$ and apply the second boundary condition

$$y(\pi) = 1 + y'(0)\pi + \int_0^{\pi} (\pi - t)u(t)dt$$

$$\pi - 1 = 1 + y'(0)\pi + \int_0^{\pi} (\pi - t)u(t)dt$$

Solve for $y'(0)$

$$y'(0) = 1 - \frac{2}{\pi} - \frac{1}{\pi}\int_0^{\pi} (\pi - t)u(t)dt$$

Then

$$\begin{aligned} y(x) &= 1 + \left[1 - \frac{2}{\pi} - \frac{1}{\pi}\int_0^{\pi} (\pi - t)u(t)dt\right]x + \int_0^{x} (x - t)u(t)dt \\ &= 1 + x - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^{\pi} (\pi - t)u(t)dt + \int_0^{x} (x - t)u(t)dt \end{aligned}$$

Now substitute this expression for $y(x)$ and $y''(x) = u(x)$ into the original differential equation

$$u + 1 + x - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^{\pi} (\pi - t)u(t)dt + \int_0^{x} (x - t)u(t)dt = x$$

$$u + 1 - \frac{2}{\pi}x - \frac{x}{\pi}\int_0^{\pi} (\pi - t)u(t)dt + \int_0^{x} (x - t)u(t)dt = 0$$

$$u = -1 + \frac{2}{\pi}x + \frac{x}{\pi}\int_0^{\pi} (\pi - t)u(t)dt - \int_0^{x} (x - t)u(t)dt$$

$$u = -1 + \frac{2}{\pi}x + \frac{x}{\pi}\int_0^{x} (\pi - t)u(t)dt + \frac{x}{\pi}\int_x^{\pi} (\pi - t)u(t)dt - \int_0^{x} (x - t)u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \left[\frac{x}{\pi}\int_0^{x} (\pi - t)u(t)dt - \int_0^{x} (x - t)u(t)dt\right] + \frac{x}{\pi}\int_x^{\pi} (\pi - t)u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \left[\int_0^{x} \frac{x}{\pi}(\pi - t)u(t)dt - \int_0^{x} (x - t)u(t)dt\right] + \int_x^{\pi} \frac{x(\pi - t)}{\pi}u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \int_0^{x} \left[\frac{x}{\pi}(\pi - t) - (x - t)\right]u(t)dt + \int_x^{\pi} \frac{x(\pi - t)}{\pi}u(t)dt$$

$$u = \frac{2}{\pi}x - 1 + \int_0^{x} \frac{t(\pi - x)}{\pi}u(t)dt + \int_x^{\pi} \frac{x(\pi - t)}{\pi}u(t)dt$$

It yields a Fredholm integral equation

$$u = \frac{2}{\pi}x - 1 + \int_0^{\pi} K(x,t)u(t)\,dt$$

with a kernel

$$K(x,t) = \begin{cases} \dfrac{t(\pi - x)}{\pi} & 0 \le t \le x \\ \dfrac{x(\pi - t)}{\pi} & x \le t \le \pi \end{cases}$$

7.7 Exercises

1. Prove part 3) of the Theorem 7.2.

2. Classify each of the following integral equations as Fredholm or Volterra integral equation, linear or non-linear, homogeneous or non-homogeneous, identify the parameter λ and the kernel $K(x,y)$:

a) $u(x) = x + \int_0^1 xyu(y)dy$

b) $u(x) = 1 + x^2 + \int_0^x (x-y)u(y)dy$

c) $u(x) = e^x + \int_0^x yu^2(y)dy$

d) $u(x) = \int_0^1 (x-y)^2 u(y)dy$

e) $u(x) = 1 + \frac{x}{4}\int_0^1 \frac{1}{x+y}\frac{1}{u(y)}dy$

3. Reduce the following integral equation to an initial value problem

$$u(x) = x + \int_0^x (y-x)u(y)dy$$

4. Find the equivalent Volterra integral equation to the following initial value problem

$$y''(x) + y(x) = \cos x \qquad y(0) = 0 \qquad y'(0) = 1$$

5. Derive the equivalent Fredholm integral equation for the following boundary value problem

$$y'' + y = x \quad x \in (0,1) \qquad y(0) = 1 \qquad y(1) = 0$$

6. Solve the following integral equations by using the successive approximation method and the resolvent method:

a) $u(x) = x + \lambda\int_0^1 xyu(y)dy$

b) $u(x) = x + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos xu(y)dy$

7. Solve the following integral equation by using the successive approximations method

$$u(x) = 1 - \int_0^x (y-x)u(y)dy$$

8. Solve the following integral equations:

a) $u(x) = sin\,2x + \int_0^x u(x-s)\,sin(s)\,ds$

b) $u(x) = x^2 + \int_0^x u'(x-s)\,e^{-as}\,ds \qquad u(0) = 0$

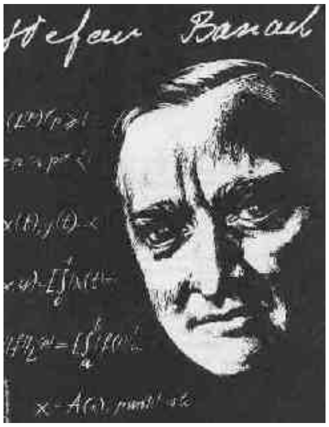
9. Using mathematical induction prove identity for iterated kernel (7.5 2):

$$K_n(x, y) \qquad = \int_G K(x, y')K_{n-1}(y', y)dy'$$

10. Using mathematical induction verify the following estimate for iterated kernels (7.5 2):

$$|K_n(x, y)| \le M^n (b-a)^{n-1}$$

11. Verify result of Example 7.4 by solving both IVP and derived integral equation.



Stefan Banach (1892 -1945)

Scottish Café

Lvov

The Scottish café in Lvov (Ukraine) was a meeting place for many mathematicians including Banach, Steinhaus, Ulam, Mazur, Kac, Schauder, Kaczmarz and others. Problems were written in a book kept by the landlord. A collection of these problems appeared later as the *Scottish Book*. R D Mauldin, *The Scottish Book, Mathematics from the Scottish Café* (1981) contains the problems as well as some solutions and commentaries.

Ivar Fredholm (1866 – 1927)

Fredholm is best remembered for his work on integral equations and spectral theory.

Find out more at: http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fredholm.html

Vito Volterra (1860 - 1940)

Volterra published papers on partial differential equations, particularly the equation of cylindrical waves. His most famous work was done on integral equations. He published many papers on what is now called 'an integral equation of Volterra type'.

Find out more at: http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Volterra.html